Unique ergodicity of branched covers of translation surfaces

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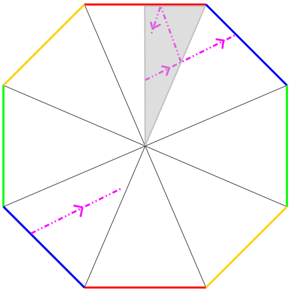
Takeaway: for almost every uniquely ergodic translation surface and for almost every slit on it, the induced branched cyclic n-cover is uniquely ergodic.

1. Translation surfaces, their applications, and the Teuchmüller flow.

Flat surfaces

A *flat surface* is a Riemann surface with a fixed holomorphic 1-form. It can be obtained by identifying the sides of a polygon in the Euclidean plane by translations, so it can be called a *translation surface* with a *vertical flow*. It is also called an *Abelian differential*.



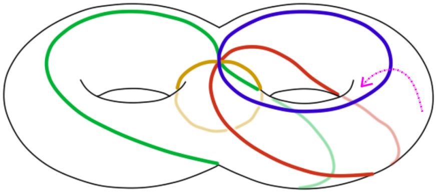


Figure: Unfolded billiard in the triangle with the angles $(\pi/2,\ 3\pi/8,\ \pi/8)$ and the flow on the resulting surface.

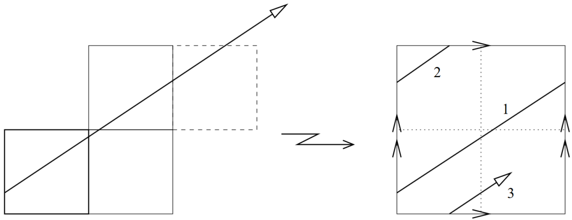
Application: Polygonal billiards

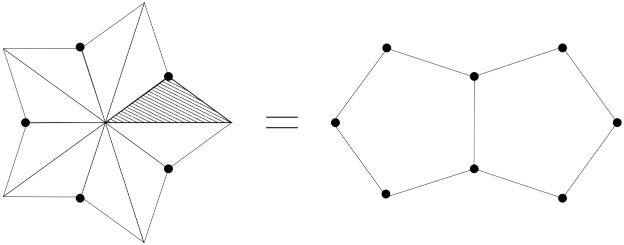
Start with:

- A polygon $P \subset \mathbb{R}^2$ with piecewise straight boundary.
- A point particle moves at unit speed inside P and reflects elastically off the sides ("angle of incidence $=$ angle of reflection").

This is the standard *planar polygonal billiard*. The key case where translation surfaces appear is when P is rational: all interior angles α_i satisfy $\alpha_i/\pi \in \mathbb{Q}$. Equivalently, the group generated by reflections in the sides of P is finite.

Application: Polygonal billiards





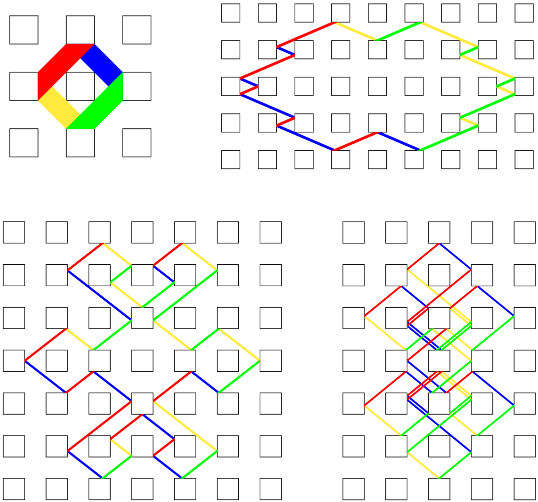
(from "Flat Surfaces" by Anton Zorich)

Application: Ehrenfest wind-tree models

In 1912, Paul and Tatiana Ehrenfest proposed the wind-tree model of diffusion to study the statistical interpretation of the second law of thermodynamics and the applicability of the Boltzmann equation. In the Ehrenfest wind-tree model, a point (“wind”) particle moves on the plane and collides with the usual law of geometric optics with randomly placed fixed square scatterers (“tree”).

The periodic Ehrenfest wind–tree model is (after unfolding) a flow on an infinite area $\mathbb{Z}^2$-cover of a compact translation surface. All the modern results on recurrence/diffusion/divergent directions are proved by passing to that translation-surface picture and using Teichmüller machinery.

Application: Ehrenfest wind-tree models



(picture by Hubert, Lelièvre, Troubetzkoy)

Application: Ehrenfest wind-tree models

Theorem (Delecroix–Hubert–Lelièvre, 2014)

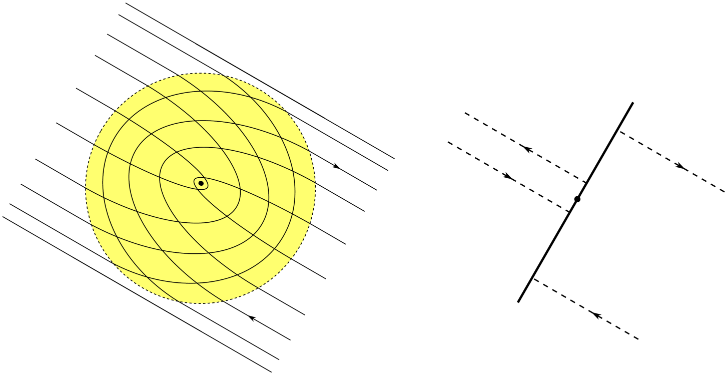
For all parameters of the obstacle, for almost all initial directions, and for any starting point, the billiard trajectory spreads in the plane with the speed $\sim t^{2/3}$*. That is,* $\lim_{t\to+\infty} \log$ *(diameter of trajectory of length* t*)* $/ \log t = \frac{2}{3}$ *(note that for round scatterers in the Sinai billiard it's* $1/2$*, which is the same diffusion rate as for a discrete random walk on* $\mathbb{Z}^2$*). The diffusion rate* $\frac{2}{3}$ *is given by the Lyapunov exponent of certain renormalizing dynamical system associated to the initial one.*

Changing the height and the width of the obstacle we get quite different billiards, but this does not change the diffusion rate!

Application: Eaton lenses

An Eaton lens is a circular lens on the plane $\mathbb{R}^2$ which acts as a perfect retroreflector, i.e. so that each ray of light after passing through the Eaton lens is directed back toward its source, see Fig. 1. More precisely, if an Eaton lens is of radius $R > 0$, then the refractive index inside the lens depends only on the distance from the center r and is given by the formula $n(x, y) = n(r) = \sqrt{2R/r - 1}$. The refractive index $n(x, y)$ is constant and equals 1 outside the lens.

Application: Eaton lenses



Light rays passing through an Eaton lens and its flat counterpart.

(Picture by Fraczek and Schmoll)

Application: Eaton lenses

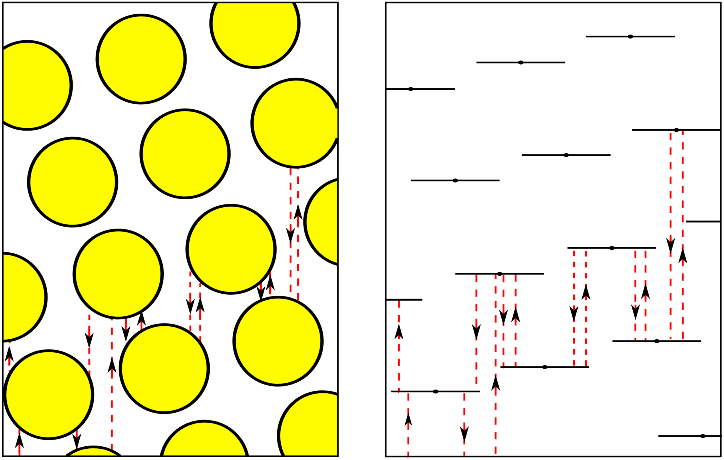
Theorem (Fraczek–Schmoll, 2014)

For almost every admissible pair (Λ, R) (lattice, radius) there exist constants $C = C(\Lambda, R) > 0$ and $\Theta = \Theta(\Lambda, R) \in \mathbb{S}^1$, such that every vertical light ray in $L(\Lambda, R)$ is trapped in an infinite band of width $C > 0$ in direction Θ.

Theorem (Fraczek–Shi–Ulcigrai, 2018)

For every admissible pair (Λ, R), for almost every direction η, there exist constants $C = C(\Lambda, R, \eta) > 0$ and $\Theta = \Theta(\Lambda, R, \eta) \in \mathbb{S}^1$, such that every light ray in direction η in $L(\Lambda, R)$ is trapped in an infinite band of width $C > 0$ in direction Θ.

Application: Eaton lenses



Eaton lense billiard.

(Picture by Fraczek and Schmoll)

Strata of translation surfaces

Let $\mathcal{M}_g$ be the Moduli space of Riemann surfaces S_g of genus g.

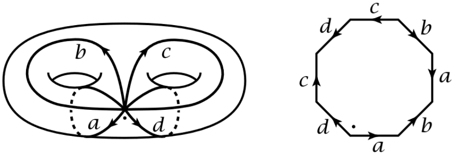
The Moduli space $\mathcal{H}_g$ of pairs (surface S_g, fixed holomorphic 1-form ω) is a bundle over $\mathcal{M}_g$.

$\mathcal{H}_g$ is not connected. It is *stratified* into components $\mathcal{H}(\alpha)$, where $\alpha = (a_1, \ldots, a_n)$, $a_1 + \ldots + a_n = 2g - 2$, are the degrees of zeroes of ω (can be thought of as conical points $S_n = \{P_1, \ldots, P_n\}$ with cone angles $2\pi(a_i + 1)$).

$\mathcal{H}(\alpha)$ a complex-analytic orbifold, not necessarily manifold, of dimension $m = 2g + n - 1$. Note that $\dim_{\mathbb{C}}(\mathcal{M}_g) = 3g - 3$, so an individual stratum doesn't always form a bundle over $\mathcal{M}_g$.

Period coordinates

For a surface $(S, \Sigma, \omega) \in \mathcal{H}(\alpha)$, denote by $\gamma_1, \ldots, \gamma_m, \ m = = 2g + n - 1$, the basis of the relative homology group $H_1(S, \Sigma; \mathbb{Z})$.



The KZ cocycle is given by the linear action of $SL_2\mathbb{R}$ on *the period coordinatess*

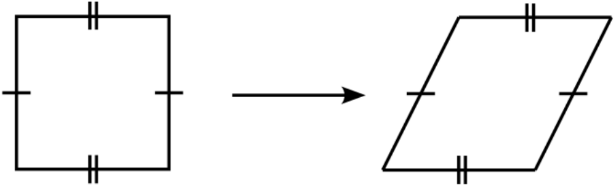
$$\int_{\gamma_1} \omega = x_1 + iy_1, \quad \ldots, \quad \int_{\gamma_m} \omega = x_m + iy_m.$$

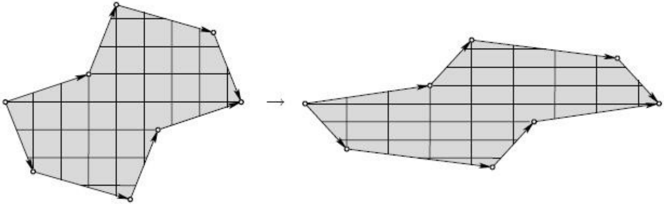
$\mathrm{SL}_2(\mathbb{R})$ acts on $\begin{pmatrix} x_1 & \ldots & x_m \\ y_1 & \ldots & y_m \end{pmatrix}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \ldots & x_m \\ y_1 & \ldots & y_m \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 & \ldots & ax_m + by_m \\ cx_1 + dy_1 & \ldots & cx_m + dy_m \end{pmatrix}.$$

Period coordinates

Example — acting by the matrix $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$:





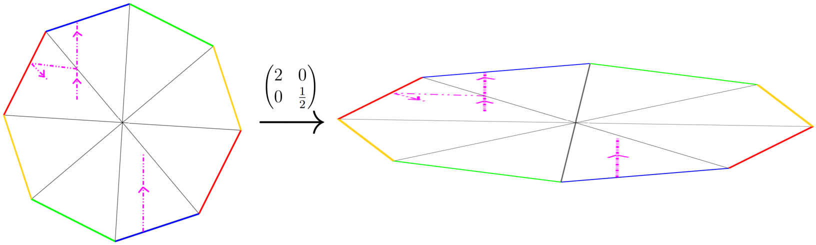
(from "Flat Surfaces" by Anton Zorich)

The Teichmüller flow

Let (X, ω) be a translation surface of area 1. In local flat coordinates $z = x + iy$ coming from ω, the *Teichmüller flow* (g_t) is defined by post-composing all charts with the linear map

$$(x, y) \longmapsto (e^t x,\ e^{-t} y).$$

Equivalently, g_t stretches the horizontal direction by e^t and contracts the vertical direction by e^{-t}, preserving area.



Ergodic properties are invariant under g_t.

2. Unique and non-unique ergodicity of translation surfaces.

Unique and non-unique ergodicity: why do we care

Birkhoff ergodic theorem (variant)

Let X be a translation surface with an ergodic probability measure m. For any $f \in L^1_X(m)$ and for m-a.e. point $P \in X$,

$$\lim_{t \to \infty} \frac{1}{T} \int_0^T f(v_t P) \,\mathrm{d}t = \int_X f(x) \,\mathrm{d}m(x).$$

- No ergodicity $\implies$ invariant sets;
- Ergodicity $\implies$ time averages $=$ space averages;
- > 2 ergodic measures $\implies$ time averages are *very* dependent on the initial point.

Minimality and periodicity of translation surfaces

A translation surface is called *minimal* if every orbit (except for those of the singularities) on it is dense.

A translation surface is said to satisfy *strict ergodicity* if every minimal direction is uniquely ergodic.

For a fixed direction on a translation surface, the surface decomposes into finitely many invariant components, each of which is either a *cylinder* of periodic orbits or a minimal component. The boundaries of the components are unions of *saddle connections* parallel to that direction.

Minimality and periodicity of translation surfaces

Theorem (Keane condition for translation surfaces)

Let (X, ω) be a translation surface and fix a direction θ. Assume that there is no saddle connection on X whose direction is θ; that is, there is no straight geodesic segment of direction θ joining two (not necessarily distinct) zeros of ω.
Then the directional flow $\{\varphi_\theta^t\}_{t\in\mathbb{R}}$ on X in direction θ is minimal: every orbit which does not hit a singularity is dense in X.

Unique and non-unique ergodicity of flat surfaces

Theorem (Masur, Veech, 1982)

Let $\mathcal{H}_1(\kappa)$ be a stratum of unit-area translation surfaces with its Masur–Veech measure μ_{MV}. Then for μ_{MV}-almost every $X \in \mathcal{H}_1(\kappa)$, the vertical translation flow on X is uniquely ergodic.

- If a translation surface X is uniquely ergodic, it is with respect to the Lebesgue measure;
- The Masur–Veech measure μ_{MV} is the Euclidean Lebesgue measure in period coordinates, restricted to the unit-area hypersurface $\mathcal{H}_1(\kappa)$ (and normalized to be finite).

Unique and non-unique ergodicity of flat surfaces

Theorem (Masur's criterion, 1992)

Let (X, ω) be a unit-area translation surface and let $(g_t)_{t\in\mathbb{R}}$ denote the Teichmüller flow. If the vertical translation flow on (X, ω) is minimal but not uniquely ergodic, then the Teichmüller orbit $\{g_t(X, \omega) : t \geq 0\}$ leaves every compact subset of the corresponding stratum in moduli space.
Equivalently: if the Teichmüller geodesic $\{g_t(X, \omega)\}$ is recurrent to a compact subset of the stratum, then the vertical flow on (X, ω) is uniquely ergodic.

Cheung-Masur (2006) construct a quadratic differential whose vertical foliation is uniquely ergodic but whose Teichmüller geodesic diverges, proving that the Masur criterion is not necessary, only sufficient.

Unique and non-unique ergodicity of flat surfaces

Theorem (Cheung–Eskin, 2004)

Let (X, ω) be an area 1 translation surface and let $(g_t)_{t\in\mathbb{R}}$ denote the Teichmüller flow. For each $t \geq 0$, let $\Delta_t(X, \omega)$ be the length of the shortest saddle connection on $g_t(X, \omega)$, and set $d(t) = -\log \Delta_t(X, \omega)$. There exists $\varepsilon > 0$ such that if there is a constant C with

$$d(t) < \varepsilon \log t + C \quad \text{for all } t \geq 1,$$

then the vertical translation flow on (X, ω) is uniquely ergodic.

Theorem (Treviño, 2014)

For each $t \geq 0$, let $\delta_t(X, \omega)$ be the length of the shortest nontrivial closed curve on $g_t(X, \omega)$. If $\int_0^\infty \delta_t(X, \omega)^2 \, dt = +\infty$, then the vertical translation flow on (X, ω) is uniquely ergodic.

Non-unique ergodicity of flat surfaces

A translation surface of genus g can have at most g ergodic invariant probability measures (Katok, Sataev).

Early examples of minimal but non-uniquely ergodic straight-line flows (and IETs) are due to Veech, Keane, Keynes–Newton, and Sataev (late 1960s–1970s).

The example of Veech is of particular interest to us.

The Veech example

- Consider a torus $[0,\,1] \times [0,\,1]$ and take two copies (sheets);
- Cut a *slit* from $(0,\,0)$ to $(0,\,\beta)$, where $0 < \beta < 1$. Identify the slit sides across two surfaces;
- On each sheet, consider the flow generated by the vector $(\alpha,\,1)$, where α is *well-approximable*: $\liminf_{n\to\infty} n \cdot d(n\alpha,\,\mathbb{Z}) = 0$.

Veech shows there exist $\alpha,\,\beta$ so that the resulting surface is minimal but not uniquely ergodic, with exactly two ergodic invariant measures μ^-, μ^+ and Lebesgue measure $\mathrm{Leb} = \frac{1}{2}\,(\mu^- + \mu^+)$.

Non-unique ergodicity of flat surfaces

- For any quadratic differential, the set of not ergodic and non-uniquely ergodic directions has Hausdorff dimension at most $\frac{1}{2}$; in particular $c(\alpha) \leq \frac{1}{2}$ for every stratum $\mathcal{H}(\alpha)$ (Masur, Masur-Smillie, 1990s).
- Cheung and Cheung–Hubert–Masur constructed examples where the set of non-ergodic directions has Hausdorff dimension $\frac{1}{2}$, showing that Masur's upper bound is sharp.
- Athreya–Chaika (2014) later proved that in $\mathcal{H}(2)$ the Masur–Smillie constant is $c(2) = \frac{1}{2}$.
- Chaika–Masur showed that the set of non-uniquely ergodic d-IETs has Hausdorff codimension $\frac{1}{2}$.

3. Slit-induced branched covers of translation surfaces and their properties.

Branched covers of translation surfaces

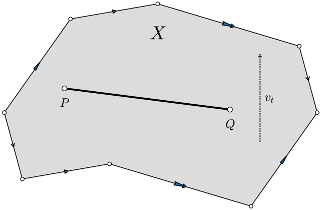


Figure: An example of a translation surface with a slit.

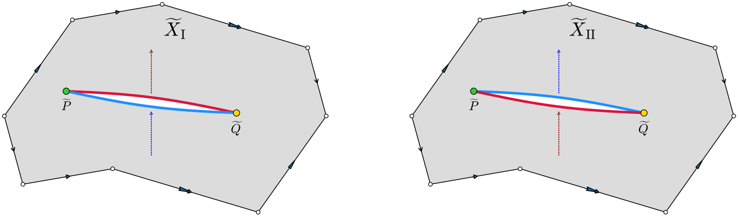


Figure: Construction of the double-cover surface $\widetilde{X}$ via its parts $\widetilde{X}_I$, $\widetilde{X}_{II}$.

Branched covers of translation surfaces

- X — a translation surface with a UE vertical flow.
- s — a slit with endpoints P and Q.
- 2 (or n) copies $(X_{\mathrm{I}}, s_{\mathrm{I}})$, $(X_{\mathrm{II}}, s_{\mathrm{II}})$ are glued into $\widetilde{X}$.
- A point on $\widetilde{X}$ can be written as (S, i). S is a position on X and $i \in \{\mathrm{I}, \mathrm{II}\}$ is the index of the copy.
- The vertical flow on $\widetilde{X}$ changes the position of the coordinate S as it would on X and switches the index i of the copy every time the flow hits either of the slits.
- I.e. there is a double (or n) cover $\pi : \widetilde{X} \to X$ branched at $\widetilde{P} = \pi^\star P$ and $\widetilde{Q} = \pi^\star Q$.
- We denote by $\widetilde{X}_{\mathrm{I}}$ and $\widetilde{X}_{\mathrm{II}}$ the natural projections of X_{I} and X_{II} resp. onto $\widetilde{X}$.

Branched covers of translation surfaces

If X is uniquely ergodic, the slit-induced cyclic branched 2-cover $\widetilde{X}$ can have no more than 2 ergodic invariant probability measures. Moreover, if the number of measures is 2, the involution $i : \widetilde{X} \to \widetilde{X}$, $i(\widetilde{X}_{\mathrm{I}}) = \widetilde{X}_{\mathrm{II}}$, exchanges their supports.

Proof: All ergodic probability measures are mutually singular and absolutely continuous with respect to the Lebesgue measure on $\widetilde{X}$ ($\mathrm{Leb}(\widetilde{X})$ projects to $\mathrm{Leb}(X)$, which is ergodic on X). Any ergodic measure must project down as $\mathrm{Leb}(X)$. The surface $\widetilde{X}$ is symmetric with respect to the involution i, so i either takes an ergodic measure to itself or exchanges it with another ergodic measure. Let μ be an ergodic probability measure on $\widetilde{X}$, then $\pi_*(\mu) = \mathrm{Leb}(X)$. Thus, the supports of μ and $i(\mu)$ must cover $\widetilde{X}$ up to Lebesgue measure 0. So there cannot exist two distinct mutually singular probability measures μ and ν on $\widetilde{X}$ such that $i(\mu) \neq \nu$ and $\pi_*(\mu) = \pi_*(\nu) = \mathrm{Leb}(X)$.

Branched covers of translation surfaces

Conjecture (Jon Chaika):

Let X be uniquely ergodic. For Lebesgue-almost every slit (P, Q, s), the cyclic n-cover slit construction is uniquely ergodic.

Why do we care? We want to understand mechanisms that produce non-unique ergodicity.

A partial result was obtained by Cheung–Eskin (2007), who gave a complete characterisation of the non-ergodic directions in the case of slit-induced double-covers of tori.

Note that all aforementioned results on unique ergodicity vary the *direction* of the vertical flow, while this conjecture fixes the direction and varies the *construction* of the surface.

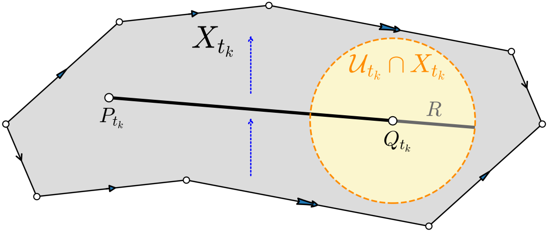
Branched covers of translation surfaces

Hubert–Ferenczi consider $\mathbb{Z}/N\mathbb{Z}$-extensions of the Veech 1969 example.

- They give a precise geometric criterion for non-minimality of the linear flow on the $\mathbb{Z}/N\mathbb{Z}$-cover X_N of the torus.
- Assuming minimality and linear recurrence, they prove unique ergodicity in special cases of $\mathbb{Z}/N\mathbb{Z}$-covers.
- Their setting is *interval exchange transformations*, and their methods come from symbolic dynamics.

4. Unique ergodicity of branched covers of translation surfaces: the statements and the sketch of the proof.

Unique ergodicity criterion



Theorem I (B.–Shuvaeva, 2025):

Let X be a uniquely ergodic surface with a slit (P, Q, s). Construct a double cover $\pi : \widetilde{X} \to X$. *Key assumption*: for only one of the slit endpoints $Q \in X$, there exists a $R > 0$ and a sequence of moments in time $\{t_k\}_{k \in \mathbb{N}_0}$, $\lim_{k\to\infty} t_k = +\infty$, such that the R-neighborhood $\mathrm{U}_{t_k} \subset X_{t_k}$ of $Q_{t_k} = g_{t_k} Q$ is an embedded disk not containing P_{t_k}. Then the surface $\widetilde{X}$ is uniquely ergodic.

Unique ergodicity criterion: Useful lemma

Let m be an ergodic invariant measure on $\widetilde{X}$. Suppose that the function f is m-integrable on $\widetilde{X}$. Denote $\mathrm{Av}_m(f) := \int_{\widetilde{X}} f(x)\,\mathrm{d}m(x)$. Let $0 < \epsilon < 1$ and $C > 0$ be some real constants. Define

$$G_{\epsilon, C} = \Bigg\{ P \in \widetilde{X} \text{ nonsingular} \;\Bigg|\; \exists \text{ erg. inv. meas.} m \text{ on } \widetilde{X} \text{ s.t.}$$

$$\forall t > C \text{ and } \forall h \in \mathrm{Lip}_1(\widetilde{X}) : \quad \left\| \frac{1}{t} \int_0^t f(v_s P)\,\mathrm{d}s - \mathrm{Av}_m(h) \right\| < \epsilon$$

$$\text{and } \left\| \frac{1}{t} \int_0^t f(v_{-s} P)\,\mathrm{d}s - \mathrm{Av}_m(h) \right\| < \epsilon \Bigg\}.$$

Lemma:

For any constant $0 < \epsilon < 1$, there exist a positive constant T_ϵ s. t.

$$\text{Lebesgue measure}(G_{\epsilon, T_\epsilon}) > 1 - \epsilon.$$

Unique ergodicity criterion: Useful lemma

Ingredients:

- Birkhoff's pointwise ergodic theorem for flows (obviously);
- Egorov's theorem (condition for the uniform convergence of a pointwise convergent sequence of measurable functions);
- Arzelà–Ascoli (compactness of the normalized 1-Lipschitz class);
- Kantorovich–Rubinstein duality (bounded-Lipschitz metric — to pass from a finite net of test functions to "all 1-Lipschitz f" in one shot).

Unique ergodicity criterion: proof sketch

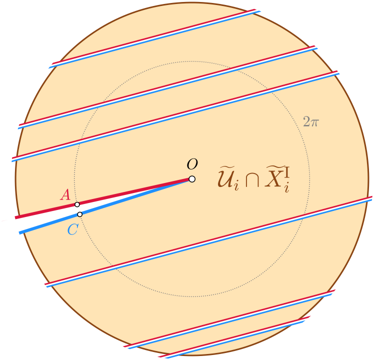
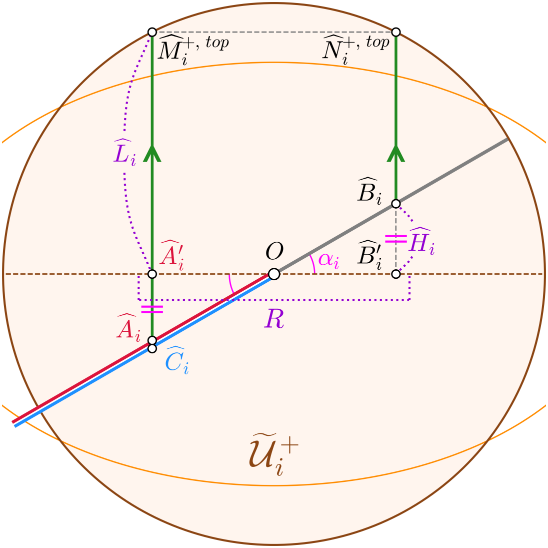


Figure: Neighborhood of the endpoint O of radius r inside $\widetilde{X}_i$.

Unique ergodicity criterion: proof sketch



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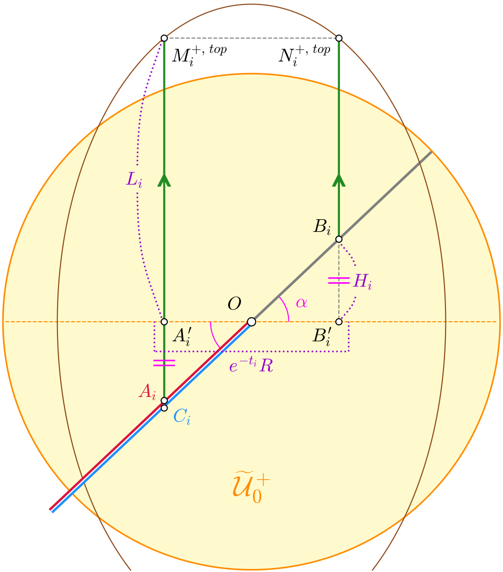
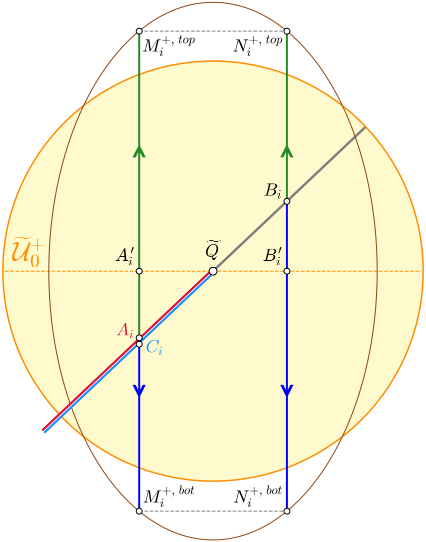


Unique ergodicity criterion: proof sketch



Corollaries

Theorem II (B.–Shuvaeva, 2025):

Let X be minimal and uniquely ergodic. Suppose there exists a sequence $t_k \to \infty$ and R such that $X_{t_k} = g_{t_k}(X)$ has diameter at least $2R$ (i.e. we can embed a disk of radius R into X). For Lebesgue-almost every slit (P, Q, s), the cyclic n-cover slit construction is uniquely ergodic.

Proof sketch: If there exists a sequence $t_k \to \infty$ and R such that we can embed a disk of radius R into X, then there exists a set S of points $P \in X$ of area at least πR^2 such that each point $P_{t_k} = g_{t_k}(P)$ lives in such embedded disk except for a finite amount of times. Therefore, unless the endpoint Q of our slit lies on a trajectory of a singularity, we can vertically flow a small disk neighbourhood of Q of radius r until its image is in S. Then we can run our argument.

Corollaries

Corollary III (B.–Shuvaeva, 2025):

Suppose that a surface X satisfies Masur's criterion of unique ergodicity. For Lebesgue-almost every slit (P, Q, s), the cyclic n-cover slit construction is uniquely ergodic.
Consequently, for almost every uniquely ergodic translation surface and for almost every slit on it, the induced branched cyclic n-cover is uniquely ergodic.

Conjecture:

All minimal uniquely ergodic translation surfaces satisfy the condition from Theorem II.

Conjecture: Reasoning

Why do we think that?Problems comes from two sources: singularities and short nontrivial loops.

- Only a measure-zero set of points is close to a singularity, so we can just exclude them as potential endpoints;

- Loops are harder. A surface that doesn't have a large embedded disk "consists" of short loops $\implies$ is a union of long thin cylinders $\implies$ can't be true because the chosen direction of the flow is non-periodic.

Useful tools: Delanay poligonalizations; boundaries of strata of translation surfaces.

Future directions

Direction A:

Construct *examples of non-uniquely ergodic branched covers*.

Recall that Veech (1969) showed that a $\mathbb{Z}_2$ skew product of an irrational rotation can be minimal and non-uniquely ergodic.

Direction B:

The conjecture of Chaika holds for quadratic differentials.

Thank you for your attention!