The Neumann–Moser dynamical system and the Korteweg–de Vries hierarchy

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The Neumann–Moser system in one time

We introduce the ∞-dimensional Neumann–Moser dynamical system in $\mathbb{C}^{3\infty} = \mathbb{C}^{\infty}\times\mathbb{C}^{\infty}\times\mathbb{C}^{1+\infty}$ with graded coordinates $U(t) = (u_2, u_4, \dots, u_{2n}, \dots)$, $V(t) = (v_3, v_5, \dots, v_{2n+1}, \dots)$, $W(t) = (w_2, w_4, \dots, w_{2n}, w_{2n+2}, \dots)$ defined by

$$\dot U^{\infty}_{\eta} = -2V^{\infty}_{\eta}, \tag{1}$$

$$\dot V^{\infty}_{\eta} = -(\eta^{-1} + w_2 - u_2)U^{\infty}_{\eta} + W^{\infty}_{\eta}, \tag{2}$$

$$\dot W^{\infty}_{\eta} = 2(\eta^{-1} + w_2 - u_2)V^{\infty}_{\eta}, \tag{3}$$

where $$U^{\infty}_{\eta} = 1 + \sum_{i=1}^{\infty} u_{2i}\eta^{i}, \qquad V^{\infty}_{\eta} = \sum_{i=1}^{\infty} v_{2i+1}\eta^{i}, \qquad W^{\infty}_{\eta} = \eta^{-1} + \sum_{i=1}^{\infty} w_{2i}\eta^{i-1}.$$

The NM system satisfies the Lax equation

$$\dot L^{\infty}_{\eta} = [L^{\infty}_{\eta},\, K^{\infty}_{\eta}], \qquad \text{where} \qquad L^{\infty}_{\eta} = \begin{pmatrix} V^{\infty}_{\eta} & U^{\infty}_{\eta} \\ W^{\infty}_{\eta} & -V^{\infty}_{\eta} \end{pmatrix}, \qquad K^{\infty}_{\eta} = \begin{pmatrix} 0 & -1 \\ -(\eta^{-1} + w_2 - u_2) & 0 \end{pmatrix},$$

The Hamiltonian of the NM system is

$$H^{\infty}_{\eta} = H^{\infty}_{\eta}(U, V, W) = -\det L^{\infty}_{\eta} = U^{\infty}_{\eta}W^{\infty}_{\eta} + (V^{\infty}_{\eta})^2 = \eta^{-1} + \sum_{i=1}^{\infty} h_{2i}\eta^{i-1}.$$

One can construct an n-stationary NM system by setting $u_{2k} = v_{2k+1} = w_{2k+2} = 0$ for all $k \geq n$.

Key properties

Lemma 1.1. The NM system is homogeneous in grading if we assume

$$\deg\partial = 1, \qquad \deg u_{2k} = \deg w_{2k} = 2k, \qquad \deg v_{2k+1} = 2k+1.$$

Lemma 1.2 (see [2, Corollary 3.7]). Let $U(t)$, $V(t)$, $W(t)$ be a solution of the NM system. For any $k > 1$, the functions w_{2k-2} and v_{2k-1} are described as differential polynomials of $u_2, \dots, u_{2k-2}$ and the constants $h_2, \dots, h_{2k}$.

Theorem 1.3 (see [2, Theorem 5.2]). We have the following recursion:

$$u_{2k} = \frac{1}{4}\ddot u_{2(k-1)} - (h_2 - \frac{3}{2}u_2)u_{2(k-1)} + \frac{1}{2}h_{2k} - \frac{1}{8}\sum_{i=1}^{k-2}\Big(4u_2u_{2(k-i)} + \dot u_{2i}\dot u_{2(k-i-1)} - 2u_{2i}\ddot u_{2(k-i-1)} + 4(h_2 - 2u_2)u_{2i}u_{2(k-i-1)}\Big).$$

Corollary 1.4. Every u_{2k}, v_{2k+1}, w_{2k} is a differential polynomial in u_2 and the constants $h_2, \dots, h_{2k}$.

Unlike in the case of σ-functional solutions [2, §7.2], we allow any values of the parameters.

Corollary 1.5 (see [1, Theorem 3.4]). The recursion in Theorem 1.3 is the Gelfand–Dikii recursion up to a constant multiplication of u_2.

Lemma 1.6 (see [2, Theorem 7.2]). For all $k \geq 1$, $\quad \dot u_{2k+2} = \frac{1}{4}u^{(3)}_{2k} + 2(u_2 - \frac{1}{2}h_2)\dot u_{2k} + \dot u_2 u_{2k}$.

Transformation to the C. Neumann system

The n-dimensional parametric C. Neumann dynamical system (1859) describes the movement of a particle on a real unitary n-sphere in $\mathbb{R}^{n+1}$ under the influence of a potential $\Gamma(t) = \sum_{i=1}^{n+1}(y_i^2 - a_i x_i^2)$ (a force that keeps the particle on this sphere):

$$\begin{cases} \dot x_k(t) = y_k(t) \\ \dot y_k(t) = -\big(\Gamma(t) + a_k\big)x_k(t), \end{cases} \qquad 1 \leq k \leq n+1, \tag{4}$$

with the vector of parameters $A = (a_1, \dots, a_{n+1}) \in \mathbb{R}^{n+1}$ such that $a_i \neq a_j$ if $i \neq j$, and with initial conditions $\sum_{i=1}^{n+1} x_i^2(0) = 1 \quad \sum_{i=1}^{n+1} x_i(0)y_i(0) = 0$.

The complexification of the C. Neumann system is well-defined. Its Poisson bracket is induced by the canonical bracket in $\mathbb{C}^{2n+2}$. The Moser Transformation [4, §3] of the C. Neumann system is a degree 2^{n+1} branched covering that transforms it into the n-stationary NM system.

Theorem 2.1 (see [2, §2] for formulas). Fix $A = (a_1, \dots, a_{n+1})$. Let $\tau = \sqrt{-1}t$, and let $(\mathbf{U}, \mathbf{V}, \mathbf{W})$ be the result of Moser transformation of the C. Neumann system. Define $U(t) = \mathbf{U}(\tau)$, $V(t) = \mathbf{V}(\tau)$, $W(t) = \mathbf{W}(\tau)$. Then $U(t)$, $V(t)$, $W(t)$ give the n-stationary NM system. We give explicit invertible matrices (whose entries are symmetric polynomials in A) that describe the Moser transformation modulo the signs of coordinate pairs (x_i, y_i). Note that $\Gamma(\tau) = w_2(t) - u_2(t)$.

Therefore all solutions of the parametric C. Neumann system can be obtained from solutions of the NM system. The same is true for integrals.

Extension of the NM system to many times

A standard method in the theory of integrable systems allows to introduce dependence of the extended NM-system on times $t_1 = t, t_3, t_5, \dots$, $\partial_i = \frac{\partial}{\partial t_i}$, where each evolutionary differential operator ∂_{2k-1} is defined by its value in $\partial_{2k-1}u_2$.

From Corollary 1.4, for the NM system in many times to be well-defined, it is enough to have

$$\partial_{2k-1}\partial_{2l-1}u_2 \equiv \partial_{2l-1}\partial_{2k-1}u_2 \;\; \forall k, l \geq 1.$$

For reasons that become apparent later, our core condition is

$$\partial_{2k-1}u_2 = \partial_1 u_{2k}.$$

Note that this is a differential polynomial in u_2 (from Corollary 1.4). We require

$$\partial_{2k-1}\partial_{2l-1}u_2 = \partial_1\partial_{2k-1}u_{2l} \equiv \partial_1\partial_{2l-1}u_{2k} = \partial_{2l-1}\partial_{2k-1}u_2 \;\; \forall k, l \geq 1.$$

Theorem 3.1. For $k, l \leq 6$, one has $\partial_{2k-1}u_{2l} \equiv \partial_{2l-1}u_{2k}$; therefore, for $1 \leq n \leq 6$, the n-stationary NM system in many times is well-defined without any extra conditions.

Conjecture A. The same holds true for any natural k, l, n.

This allows for purely computational solutions for any base function u_2 and any parameters.

This also radically simplifies the Mumford system in the form by Vanhaecke [6, §VI] (also [5, §3]).

The NM system as a model for the KdV hierarchy

Let $g = g(t = t_1, t_3, t_5, \dots)$. Define the (formal) Lenard operator

$$\Lambda[g] = \tfrac{1}{4}\partial_t^2 - u - \tfrac{1}{2}g_t\,\partial_t^{-1}, \quad \text{where } \partial_t^{-1} \text{ is the zero-mean antiderivative.}$$

Let $\{G_{2k}(g)\}_{k\geq 0}$ be differential polynomials in g determined by

$$\partial_t G_{2k+2} = \Lambda[g]\big(\partial_t G_{2k}\big), \qquad k \geq 0,$$

with seeds $G_0 \equiv \frac{1}{2}$ and $G_2 \equiv -\frac{1}{2}g$. The KdV hierarchy is the infinite family of compatible flows

$$\partial_{t_{2k+1}}G_2 = \partial_t G_{2k+2}(g), \qquad k = 0, 1, 2, \dots$$

The first nontrivial flow ($k = 1$) is the KdV equation

$$4\,g_{t_3} = 6\,g\,g_t - g_{ttt}.$$

We call a solution n-stationary if $\partial_{t_{2j+1}}g \equiv 0$ for all $j \geq n$ but $\partial_{t_{2n-1}}g \not\equiv 0$ (G_{2j+2} is 0 for $j \geq n$).

The KdV equation describes the propagation of waves in media where the effects of nonlinearity and dispersion are balanced. In 1967, Gardner, Green, Kruskal, and Miura showed that the eigenvalues of the Schrödinger operator

$$L = -\frac{\partial^2}{\partial t^2} + g,$$

where $g = u(t, t_3)$ is a solution of the KdV-equation, are integrals of the KdV-equation. In 1968, Lax showed that the KdV-equation is equivalent to the equation

$$\partial_{t_3}(L) = [\partial_t^3 - \frac{3}{2}u\partial_t - \frac{3}{4}\dot u,\, L].$$

Theorem 4.1 (see [2, Theorem 7.2]). Let $U(t)$, $V(t)$, $W(t)$ form an n-stationary NM system. Then $u_2, \dots, u_2n$ are differential polynomials in $\Gamma = w_2 - u_2 = h_2 - 2u_2$ such that

$$\dot u_2 = -\frac{1}{2}\dot\Gamma, \qquad \dot u_{2i+2} = \frac{1}{4}u^{(3)}_{2i} - \Gamma\dot u_{2i} - \frac{1}{2}\dot\Gamma u_{2i} \quad \text{for all} \quad 1 \leq i \leq n-1,$$

$$0 = u^{(3)}_{2n} - 4\Gamma\dot u_{2n} - 2\dot\Gamma u_{2n}.$$

Consequently, solving the KdV hierarchy is equivalent to solving the NM system in many times.

Quantization of the NM system in many times

Quantization of an integrable model promotes its Lax data to operator-valued matrices so that transfer matrices commute and generate quantum Hamiltonians. Buchstaber & Mikhailov [3] propose a novel approach to quantizing stationary flows of the KdV hierarchy, known as the N-Novikov equations, and obtain quantum Novikov N-hierarchies for $N \leq 4$.

Let $\mathcal{A}_0 = \big(\mathbb{C}[s_0, s_1, \dots], D\big)$ with $D(s_k) = s_{k+1}$, $s_0 = s$ being some function. Set the Schrödinger Lax operator $L = D^2 - s_0$ and its formal square root

$$\mathcal{L} = D + \sum_{n\geq 1} I_{1,n}D^{-n}, \qquad \mathcal{L}^2 = L.$$

Define Gelfand–Dikii densities $\rho_{2k} = \operatorname{res}\mathcal{L}^{2k-1}$ ($\rho_0 = 1$). The KdV flows are

$$\partial_{t_{2k-1}}s_0 = X_{2k-1}(s_0) = -2D(\rho_{2k}), \qquad k \geq 1.$$

Let $\mathcal{B}_0 = (\mathbb{C}\langle s_0, s_1, \dots\rangle, D)$ be the free differential algebra. The same pseudodifferential calculus yields non-commutative densities $\varrho_{2k} = \operatorname{res}\mathcal{L}^{2k-1}$ (monomials do not commute). Set

$$\mathcal{F}_{2N+2} = \varrho_{2N+2} + \sum_{k=0}^{N-1}\alpha_{2(N-k+1)}\varrho_{2k}, \qquad \mathcal{I}_N = (\mathcal{F}_{2N+2}, D\mathcal{F}_{2N+2}, \dots),$$

and the quotient $\mathcal{B}_N = \mathcal{B}_0/\mathcal{I}_N \cong \mathbb{C}\langle s_0, \dots, s_{2N-1}\rangle$ carries induced commuting derivations (the finite non-commutative N-Novikov hierarchy). Non-commutative first integrals live in the cyclic quotient $\mathcal{B}_N/\mathrm{Span}[\mathcal{B}_N, \mathcal{B}_N]$ and are expressed via a bilinear form σ adapted to the non-commutative residue (so that $D(\sigma) = \operatorname{res}[\cdot,\cdot]$). The derivations $\partial_{t_{2k-1}}$ induce derivations $\mathcal{D}_{2k-1}$ of $\mathcal{B}_N$.

Introduce graded parameters $\mathbf{q}$ and work over $\mathcal{B}_N(\mathbf{q})$. A two-sided homogeneous ideal $\mathcal{Q}_N \subset \mathcal{B}_N(\mathbf{q})$ is a quantisation ideal if (i) it is Poincaré–Birkhoff–Witt (PWB; ordered monomials project to a nondegenerate basis of $\mathcal{C}_N := \mathcal{B}_N(\mathbf{q})/\mathcal{Q}_N$), and (ii) it is invariant under the N-Novikov derivations. The PBW property provides a well-defined normal ordering and yields a consistent quantum N-Novikov hierarchy on $\mathcal{C}_N$.

Within a cyclic Frobenius algebra framework [3, §2.2], quantum flows take the Heisenberg form

$$\partial_{t_{2k-1}}s_\ell = \frac{i}{\hbar}\big[\widehat{\mathcal{H}}_{2k-1,\,2N+1},\, s_\ell\big] \mod \mathcal{Q}_N,$$

and the first N Hamiltonians commute. For $N \leq 4$, the normally ordered quantum N-Novikov equations coincide (as formulas) with the classical commutative ones.

Example $N = 1$. The ideal $\mathcal{Q}_1 \subset \mathcal{B}_1(\mathbf{q})$ is $\mathcal{D}$-invariant iff $s_0s_1 - s_1s_0 = i\hbar$, where $\hbar$ is an arbitrary parameter. The quantum Novikov's 1-equation can be written in the Heisenberg form

$$\partial_{t_1}s_0 = s_1 = \frac{i}{\hbar}[\mathfrak{H}_{3,3}, s_0], \qquad \partial_{t_1}s_1 = 3s_0^2 + 8\alpha_4 = \frac{i}{\hbar}[\mathfrak{H}_{3,3}, s_1],$$

where the Hamiltonian operator is $\mathfrak{H}_{3,3} = \frac{1}{2}s_1^2 - s_0^3 - 8\alpha_4 s_0$ and $[s_1, s_0] = i\hbar$.

Example $N = 2$. The quantum Novikov's 2-equation can be written in the Heisenberg form

$$\partial_{t_1}s_k = \frac{i}{\hbar}[\mathfrak{H}_{3,5}, s_k] = \begin{cases} s_{k+1}, & 0 \leqslant k \leqslant 2, \\ 32\alpha_6 - 16\alpha_4 s_0 + 5s_1^2 + 10s_2u - 10s_0^3, & k = 3. \end{cases}$$

The quantum dynamical system $\mathcal{C}_2$, corresponding to the derivations $\mathcal{D}_3$, can be written in the Heisenberg form

$$4\partial_{t_3}s_k = \frac{i}{\hbar}[\mathfrak{H}_{5,5}, s_k] = \mathcal{D}^k(s_3 - 6s_1s_0).$$

Conjecture B. The same N-stationary quantum KdV hierarchy can be obtained by quantizing the N-stationary NM system using the same methods for all N.

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