Semisimplicity and Rigidity of the Kontsevich–Zorich Cocycle on Products of Strata

THE UNIVERSITY OF CHICAGO

Polina Baron

Translation surfaces

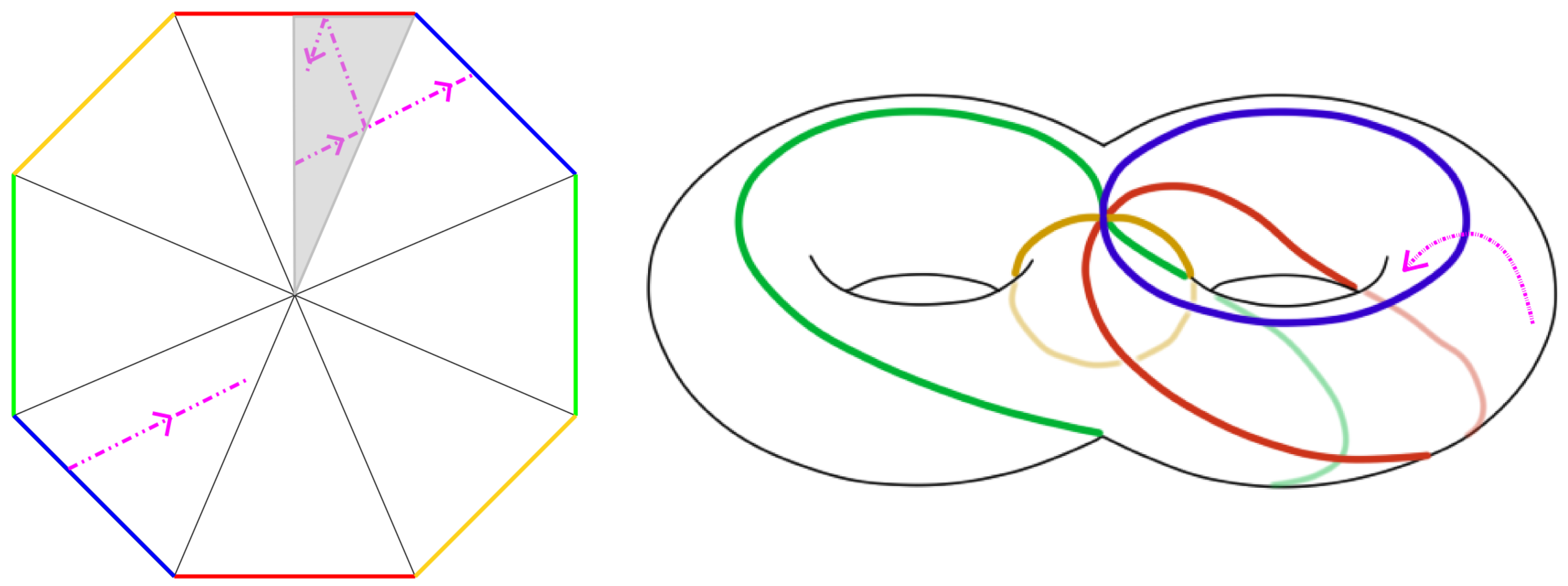

Figure 1. Unfolded billiard in the triangle with the angles $(\pi/2,\ 3\pi/8,\ \pi/8)$ and the flow on the resulting surface.

Definition. A *flat surface* is a Riemann surface with a fixed holomorphic 1-form. It can be obtained by identifying the sides of a polygon in the Euclidean plane by translations, so it can be called a *translation surface*. It is also called an *Abelian differential*.

Let $\mathcal{M}_g$ be the Moduli space of Riemann surfaces S_g of genus g. The Moduli space $\mathcal{H}_g$ of pairs (surface S_g, fixed holomorphic 1-form ω) is a bundle over $\mathcal{M}_g$. It is *stratified* into components $\mathcal{H}(\alpha)$, where $\alpha = (a_1, \dots, a_m)$, $a_1 + \dots + a_m = 2g-2$, are the degrees of zeroes of ω (can be thought of as conical points $\Sigma_m = \{P_1, \dots, P_m\}$ with cone angles $2\pi(a_i+1)$). $\mathcal{H}(\alpha)$ is a complex-analytic orbifold, not necessarily a manifold, of dimension $2g+m-1$. Note that $\dim_{\mathbb{C}}(\mathcal{M}_g) = 3g-3$, so an individual stratum doesn't always form a bundle over $\mathcal{M}_g$.

The study of translation surfaces has applications to the study of rational billiards (see Figure 1 for an example) and interval exchange transformations.

For more details, see [5] and [4].

The Kontsevich–Zorich cocycle

For a surface $(S,\ \Sigma,\ \omega) \in \mathcal{H}(\alpha)$, denote by $\gamma_1, \dots, \gamma_k$, $k = 2g+m-1$, the basis of the relative homology group $H_1(S,\ \Sigma;\ \mathbb{Z})$. The KZ cocycle is given by the linear action, *modulo regluings*, of $SL_2\mathbb{R}$ *on the periods*

$$\int_{\gamma_1} \omega = x_1 + iy_1, \quad \dots, \quad \int_{\gamma_k} \omega = x_k + iy_k,$$

$\mathrm{SL}_2(\mathbb{R})$ acts on the periods by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \dots & x_k \\ y_1 & \dots & y_k \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 & \dots & ax_k + by_k \\ cx_1 + dy_1 & \dots & cx_k + dy_k \end{pmatrix}.$$

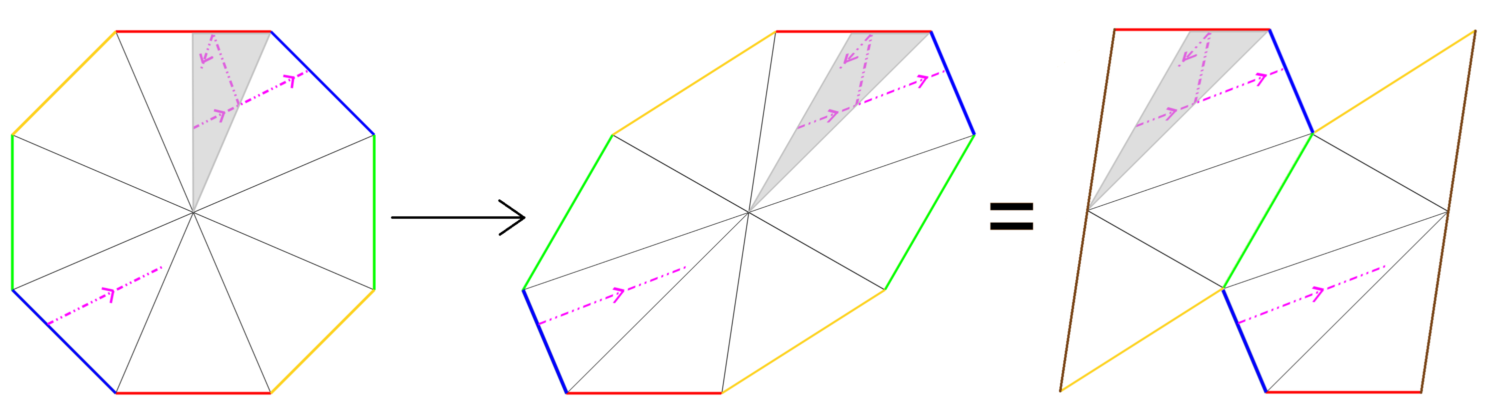

Figure 2. Action of the matrix $\begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & 1 \end{pmatrix}$ On the surface from the previous example.

Since elements of $\mathcal{H}(\alpha)$ are invariant under the action of the group $\mathrm{MCG}_g/\mathfrak{T}_g$ (Mapping Class Group factored by its Torelli subgroup), the KZ cocycle on $\mathcal{H}(\alpha)$is **highly nontrivial**. Namely, the actual fiber identification looks closer to

$$x = \begin{pmatrix} x_1 & \dots & x_k \\ y_1 & \dots & y_k \end{pmatrix} \to hx = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & \dots & x_k \\ y_1 & \dots & y_k \end{pmatrix} A(g,x),$$

where $A(h,x) \in \mathrm{Sp}(2g,\mathbb{Z}) \ltimes \mathbb{R}^{m-1}$ is the change of basis one needs to perform to return the point hx to the fundamental domain, given by the monodromy of the Gauss-Manin connection restricted to the orbit of $SL(2,\mathbb{R})$.

Products of strata of translation surfaces

Products of strata are interesting.

1. $(T,\ X,\ \mu)$ is weak mixing $\Leftrightarrow$ $(T \times T,\ X \times X,\ \mu \times \mu)$ is ergodic.
2. Compactifications of orbit closures on strata of differentials have nodal curves as boundaries of linear subvarieties (see [1]).

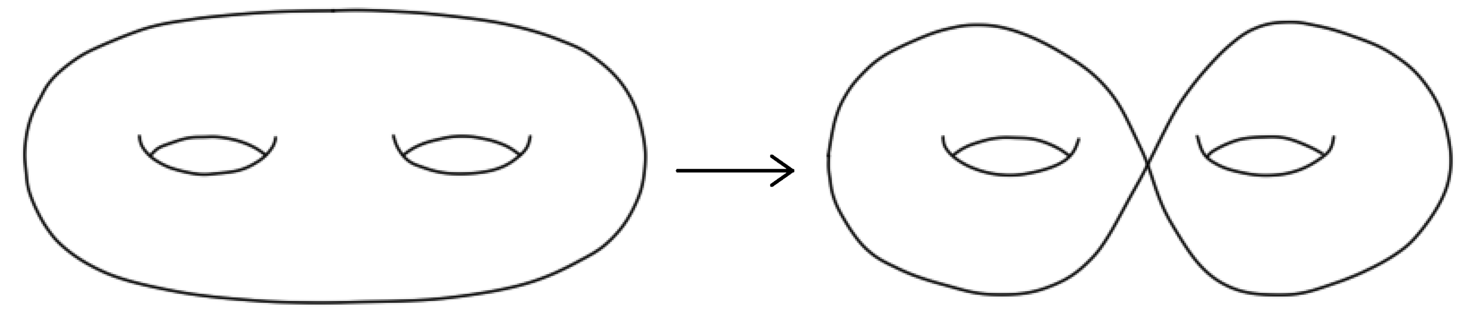

Figure 3. Deformation of a genus 2 surface into a nodal genus $(1,1)$ surface at the boundary of the compactification of $\mathcal{H}_2$.

Lemma (after Goursat's lemma): If $G \subsetneq \prod_{i=1}^n SL_2\mathbb{R}$ projects on each of its factors surjectively but does not decompose into a product, then G is a conjugate of the diagonal subgroup:

$$G = \{g \times h_2^{-1} g h_2 \times \dots \times h_n^{-1} g h_n \mid g \in SL_2\mathbb{R},\ \ h_2, \dots, h_n \text{ are fixed}\}.$$

Hodge structures and their variations

Hodge structures generalise the idea behind the Hodge decomposition theorem for complex cohomology:

$$H^r(X, \mathbf{C}) = \bigoplus_{p+q=r} H^{p,q}(X), \quad \text{where } \overline{H}^{p,q}(X) = H^{q,p}(X).$$

Definition. A *weight n Hodge structure on $E_{\mathbb{R}}$* is the orthogonal decomposition of (the complexification) $E_{\mathbb{C}}$ into

$$E_{\mathbb{C}} = \bigoplus_{p+q=n} E_{\mathbb{C}}^{p,q} \quad \text{such that} \quad E_{\mathbb{C}}^{p,q} = \overline{E_{\mathbb{C}}^{q,p}} \ \ \forall\, p+q=n.$$

The *Hodge filtration* is

$$F_i = \bigoplus_{p \geq i} E_{\mathbb{C}}^{p,q}.$$

The KZ cocycle is a *variation of Hodge structures* over a stratum $\mathcal{H}(\alpha)$: a family of Hodge structures parameterised by a complex manifold/orbifold (the stratum). The same notion can be generalised to products of strata.

Invariant subbundles

Let μ be an ergodic G-invariant measure on a product of $n \geq 1$ (here, $G \prod_{i=1}^n SL_2\mathbb{R}$ project ssurjectively on each factor). A subbundle V is *G-invariant* if it is measurable and invariant under the parallel transport along a.e. G-orbit.

Problem: $E_{\mathbb{C}}^{p,q}$ are typically not G-invariant.

Example: The tautological plane $(\langle Re(\omega),\ Im(\omega)\rangle \in H^1(S,\ \mathbb{R})$ at each $(S,\ \omega))$ is not flat unless the manifold is a Teichmüller curve or "rank 1".

Question: How do Hodge decompositions of G-invariant subbundles look like?

Applications:

1. Measure rigidity: any measurable $SL_2\mathbb{R}$-invariant bundle has to be real-analytic (Filip).
2. Affine invariant submanifolds (linear equations in local period coordinates) in strata are algebraic varieties (Filip).
3. Lyapunov exponents.
   - Zero Lyapunov exponents (Eskin, Kontsevich, Zorich, and many others);
   - Formula for the sum of Lyapunov exponents in terms of degrees of Hodge bundle (Forni, Matheus, Zorich, and many others).

Product of strata highlights

1. Strata $\mathcal{H}(\alpha_i)$, $1 \leq i \leq n$, can be different. They can even be of different genera g_i.
2. The set of ergodic G-invariant measures is bigger than the set of ergodic $\prod_{i=1}^n SL_2\mathbb{R}$-invariant measures. We might not even know what our ergodic G-invariant measure μ looks like.

Deligne semisimplicity for products of strata

Theorem I (B.): Let $\mathcal{H}_n = \mathcal{H}(\alpha_1) \times \dots \times \mathcal{H}(\alpha_n)$ be a product of n strata. Let $G \subset \prod_{i=1}^n SL_2\mathbb{R}$. Any G-invariant subbundle V has a decomposition into G-invariant components that are Hodge-orthogonal and respect the Hodge structure:

$$V = \bigoplus V_i, \quad V_i = \bigoplus (V_i \cap E_{\mathbb{C}}^{p,q}).$$

(This result is a generalisation of the Deligne semisimplicity result for the case of one stratum by S. Filip, 2013 [3].)

Main ideas:

1. Relationship between the Gauss–Manin connection (flat structure) and the Hodge connection (direct sum of connections of the decomposition) $\Rightarrow$ flat along $SL_2\mathbb{R}$ orbits sections have flat $(p,\ q)$-components;
2. The *algebraic hull* of the KZ cocycle is reductive (any invariant subspace has invariant orthogonal compliment);
3. Therefore, we can decompose the subbundle, constructing the variation of Hodge structures iteratively.

Rigidity for products of strata

In the setup of the previous theorem, consider an *affine invariant submanifold* $\mathcal{M}$ (an immersed, each point has a neighbourhood whose image is locally defined by real linear equations in period coordinates). A tensor construction V of its KZ cocycle has a decomposition into G-invariant subbundles (as per the previous theorem).

Theorem II (B.): These subbundles vary polynomially in the directions transverse to the G-action.

(This result is a generalisation of the analogous result for the case of one stratum by S. Filip, 2013 [3].)

Main ideas:

1. Use Hodge-orthogonality of the decomposition from Theorem I and results from [2] to establish real-analyticity along a.e. stable or unstable leaf for each V_i.
2. The dependence along leaves must be polynomial based on contraction properties of the Teichmüller flow.
3. Prove joint polynomiality in all directions.

References

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